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RESEARCH MEMORANDUM

CHARACTERIZING DISTRIBUTIONS BY QUANTILE MEASURES

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Abstract. Modelling an empirical distribution by means of a simple theoretical distribution is an interesting issue in applied statistics. A reasonable first step in this modelling process is to demand that measures for location, dispersion, skewness and kurtosis for the two distributions coincide. Up to now, the four measures used hereby were based on moments.

In this paper measures are considered which are based on quantiles. Of course the four values of these quantile measures do not uniquely determine the modelling distribution. They do, however, within specific systems of distributions, like Pearson's or Johnson's.

This opens the possibility of modelling - within a specific system - an empirical distribution by means of quantile measures. Since moment-based measures are sensitive for outliers, this approach may lead to a better fit.

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1. A quantile measure for kurtosis

Consider a random variable $\underline{x}$ with mean $\mu = E(\underline{x})$ and central moments

$$\mu_i = E(\underline{x}-\mu)^i , \quad i = 2,3,...$$

The (very familiar) moment-based measures for location, dispersion, skewness and kurtosis now are

- the mean μ
- the variance μ_2
- the third standardized moment $\beta_1 = \mu_3/\mu_2^{3/2}$
- the fourth standardized moment $\beta_2 = \mu_4/\mu_2^2$

They all exist provided $E(\underline{x}^4) < \infty$.

For the first three measures quantile-based alternatives are well-known. Defining quartiles Q_i by

$$P(\underline{x} < Q_i) \leq i/4 , \quad P(\underline{x} > Q_i) \leq 1 - i/4$$

for i = 1,2,3, they are given by

- the median $Q = Q_2$
- the half interquartile range $R = (Q_3-Q_1)/2$
- Bowley's skewness measure $S = (Q_3-2Q_2+Q_1)/(Q_3-Q_1)$

provided that $Q_3 \neq Q_1$. Moors (1986, 1988) presented a new interpretation of kurtosis as well as a quantile-based alternative for β_2. Define octiles E_i by

$$P(\underline{x} < E_i) \leq i/8 , \quad P(\underline{x} > E_i) \leq 1 - i/8$$

for i = 1,2,...,7. Then the quantile measure T for kurtosis reads

$$T = \frac{(E_7-E_5) + (E_3-E_1)}{E_6 - E_2}$$

provided that $E_6 \neq E_2$. Note that T is much less sensitive for outliers than β_2; it can be calculated by graphical means. Furthermore, T exists even for distributions without finite moments; e.g. T = 2 for the Cauchy distribution.

The quartet (Q,R,S,T) can be seen as an alternative to $(\mu,\mu_2,\beta_1,\beta_2)$. Like β_1 and β_2, S en T remain unchanged under linear transformations: these four quantities are location-scale-invariant. This is the main reason why in the sequel attention is focussed on the pair (S,T).

2. The Pearson system of distributions

The Pearson system of distributions is based on the following differential equation:

$$\frac{d \log f(x)}{dx} = \frac{x}{B_0 + B_1 x + B_2 x^2}$$

Solutions f are densities within the Pearson system. These solutions depend on the zeros of the denominator or - more specifically - on the quantity

$$K = B_1^2/(4B_0B_2)$$

For $K < 0$, $0 < K < 1$, $K > 1$ three main types of distributions arise; the limiting cases $K = 0$, $K = 1$ or $K \to \infty$ lead to transition types. Table 1 shows the details.

Table 1. Outline of the Pearson system.

	Name	Type	Density*	Range	Parameters
$K < 0$	Beta 1	I	$x^{p-1}(1-x)^{q-1}$	$[0,1]$	$p,q > 0$
$K = 0$	Student	VII	$(1+x^2/n)^{-(n+1)/2}$	$\mathbb{R}$	$n > 0$
$0 < K < 1$	Arctan	IV	$(1+x^2)^{-m}\exp[v \arctan x]$	$\mathbb{R}$	$m > 1/2$, $v \in \mathbb{R}$
$K = 1$	Inverse gamma	V	$x^{-(\rho+1)}e^{-1/x}$	$\mathbb{R}^+$	$\rho > 0$
$K > 1$	Beta 2	VI	$x^{p-1}/(x+1)^{p+q}$	$\mathbb{R}^+$	$p,q > 0$
$K \to \infty$	Gamma	III	$x^{\rho-1}e^{-x}$	$\mathbb{R}^+$	$\rho > 0$

* up to normalizing constant.

The column 'Type' contains the Roman numbers originally used by Pearson to indicate the different classes of distributions. (The missing type II consists of the symmetrical Beta 1 distributions.) Location-scale parameters have been deleted from the densities, as well as normalizing constants. See for details about all this Stuart & Ord (1987), p. 210 ff.

Since the Arctan distributions are relatively unfamiliar, Figures 1 and 2 show some densities for type IV.

Figure 1. Densities of Pearson type IV; v = 0.5.

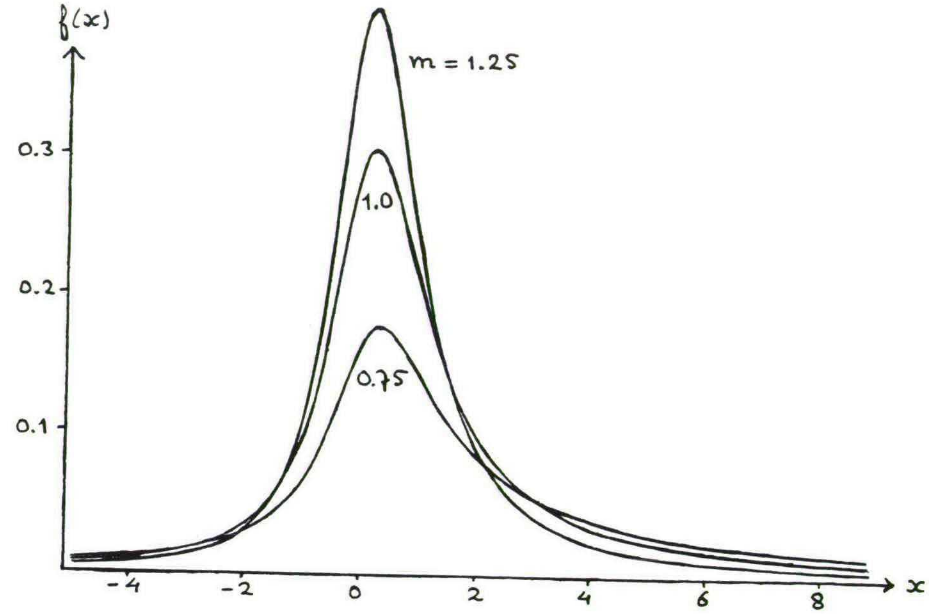
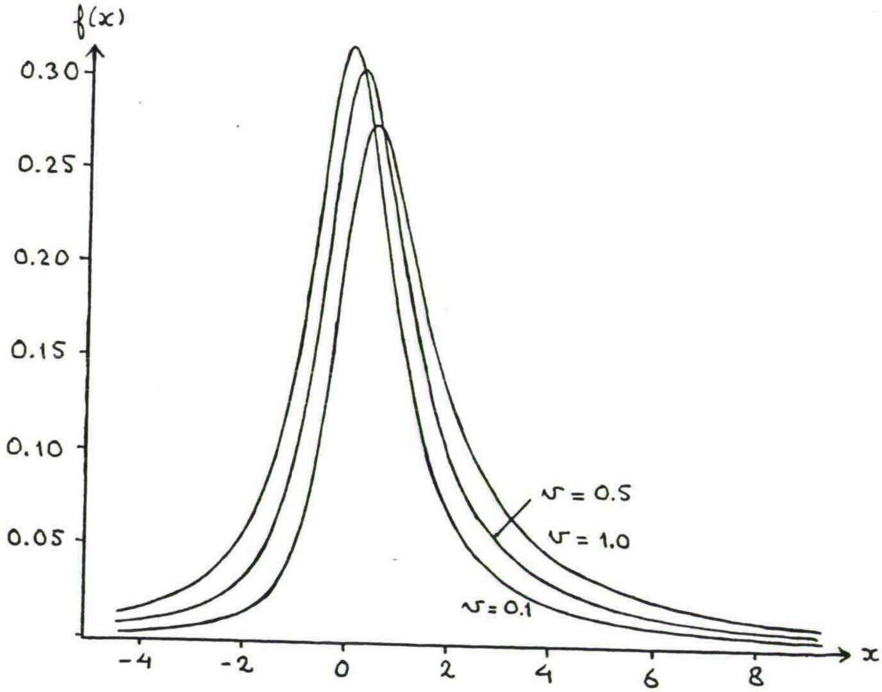
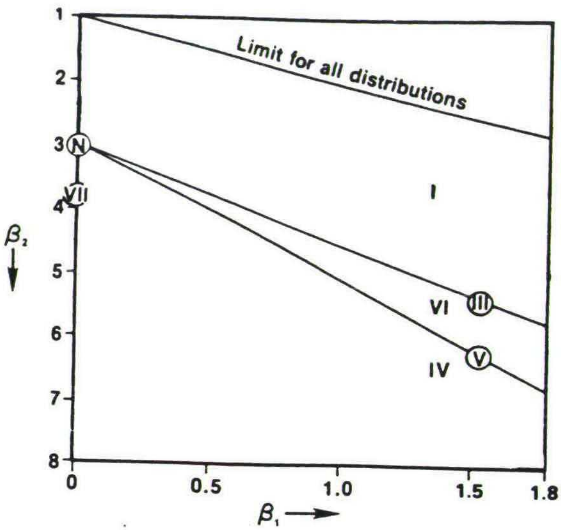


Figure 2. Densities of Pearson type IV; m = 1.



For our purposes, the main property of the Pearson system is that any (location-scale-free) distribution has a unique pair of values for the measures β_1 and β_2. In other words, there is a one-one relation between the distributions in Table 1 and points in the (β_1,β_2)-plane. Figure 3 shows this relation; the (symmetric) half-plane with $\beta_1 < 0$ is omitted. Compare Stuart & Ord (1987), p. 211.

Figure 3. The (β_1,β_2)-plane for the Pearson system.



The main types appear to occupy separate parts of the plane. Transition type III corresponds to the straight line $2\beta_2 - 3\beta_1 = 6$; the set of type V distributions is slightly curved. The Pearson system leaves unoccupied the upper righthand corner above the line $\beta_2 - \beta_1 = 1$.

In summary: all distributions in the Pearson system can be characterized by the quartet $(\mu_1,\mu_2,\beta_1,\beta_2)$. Hence, the empirical counterpart $(\bar{x},s^2,b_1,b_2)$ of this quartet corresponds with exactly one distribution within the Pearson system. This distribution can be taken as a simple model for the empirical distribution, based on moment measures. Note that from the quartet the parameters of the corresponding Pearson distribution can be found analytically.

3. Characterizing the Pearson system by quantile measures

In this section the behaviour is investigated of the pair (S,T) for distributions in the Pearson system. First of all, convergence of distributions implies convergence of the pair (S,T). In particular, the (S,T)-values of a transition type arise as limits of the (S,T)-values of main type distributions. This statement will be proved here for one limiting case only: I → III.

Starting point is the following limiting property of the gamma function Γ:

$$\lim_{n\to\infty} \frac{\Gamma(n+\rho)}{n^{\rho}\Gamma(n)} = 1$$

which can be proved by means of Stirling's formula. Let the distribution of a random variable $\underline{x}$ be denoted by $\mathcal{L}(\underline{x})$.

Lemma 1. If $\underline{x}_n \sim \mathrm{Be}(\rho,n)$,

$$\mathcal{L}(n\underline{x}_n) \to \Gamma(1,\rho)$$

holds for $n \to \infty$.

Proof. Let q_n be the density of $n\underline{x}_n$ and p that of $\Gamma(1,\rho)$; then it is sufficient to show that $q_n \to p$ pointwise if $n \to \infty$. Now $\underline{x}_n$ has density

$$B(\rho,n)^{-1}x_n^{\rho-1}(1-x_n)^{n-1},$$

where $B(\rho,n) = \Gamma(\rho)\Gamma(n)/\Gamma(\rho+n)$. For $\underline{y} = n\underline{x}_n$ it follows:

$$\begin{aligned} q_n(y) &= \frac{1}{nB(\rho,n)}\, x_n^{\rho-1}(1-x_n)^{n-1}, \quad 0 < x_n < 1, \quad \text{with } x_n = y/n, \\ &= \frac{1}{nB(\rho,n)}\, (\tfrac{y}{n})^{\rho-1}(1-\tfrac{y}{n})^{n-1}, \quad 0 < y < n, \\ &= \frac{1}{n^{\rho}B(\rho,n)}\, y^{\rho-1}(1-\tfrac{y}{n})^{n-1}. \end{aligned}$$

Now the limits

$$\lim_{n\to\infty}(1-y/n)^n = e^{-y}\ , \qquad \lim_{n\to\infty} \frac{1}{n^{\rho}B(\rho,n)} = \frac{1}{\Gamma(\rho)}$$

imply

$$\lim_{n\to\infty} q_n(y) = \frac{1}{\Gamma(\rho)}\, y^{\rho-1}e^{-y} = p(y)$$

which proves the lemma. □

Theorem 1. Let (S_n,T_n) and (S_o,T_o) denote the quantile measures of skewness and kurtosis for $Be(\rho,n)$ and $\Gamma(1,\rho)$, respectively. Then

$$\lim_{n\to\infty} S_n = S_o\ , \qquad \lim_{n\to\infty} T_n = T_o$$

Proof. Since S and T are invariant under linear transformations, they are identical for $\underline{x}_n$ and $n\underline{x}_n$. Now, the theorem is an immediate result of the Lemma. □

So the conclusion is, that smooth transitions between the various types exist in the (S,T)-plane - just as in the (β_1,β_2)-plane.

Quantiles for the type I, III, V and VI can be found directly by means of the statistical computer package SAS, while for VII Smirnov (1961) was used. For type IV a special program was written which uses numerical integration. This led to outcomes that differed slightly from the values in Johnson et al (1963). Hence, another program was written, which confirmed our previous results. Table 2 is a brief abstract from the extensive results in Wagemakers (1991).

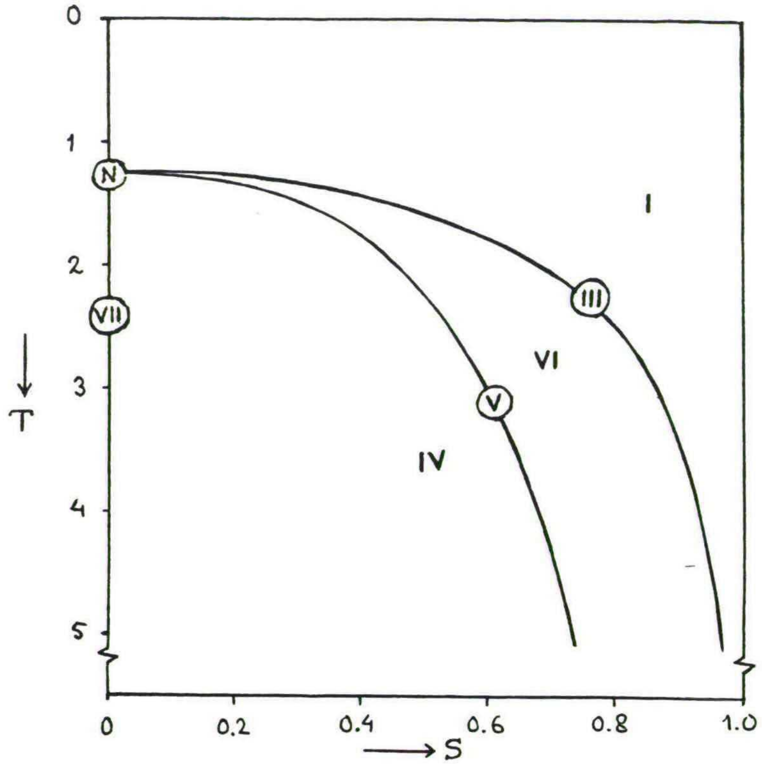
Table 2. Octiles and (S,T)-values for the Pearson system.

Type	p_1	p_2	E_1	E_2	E_3	E_4	E_5	E_6	E_7	S	T
I	0.30	0.3	0.007	0.068	0.236	0.500	0.764	0.932	0.993	0.000	0.529
I	0.30	1.2	0.001	0.008	0.030	0.079	0.168	0.317	0.557	0.541	1.349
I	0.30	2.4	0.000	0.003	0.013	0.035	0.076	0.153	0.300	0.579	1.581
I	1	1	0.125	0.250	0.375	0.500	0.625	0.750	0.875	0.000	1.000
I	1	4	0.033	0.069	0.111	0.159	0.218	0.293	0.405	0.197	1.190
I	1	8	0.017	0.035	0.057	0.083	0.115	0.159	0.229	0.230	1.244
III	0.30		0.001	0.007	0.027	0.073	0.165	0.343	0.740	0.606	2.004
III	0.75		0.058	0.153	0.283	0.454	0.688	1.034	1.650	0.317	1.347
III	1		0.134	0.288	0.470	0.693	0.981	1.386	2.079	0.262	1.306
III	5		2.617	3.369	4.020	4.671	5.390	6.274	7.599	0.104	1.243
IV	0.70	0.1	-12.11	-2.070	-0.479	0.347	1.444	4.542	26.37	0.269	5.530
IV	0.70	0.5	-1.862	-0.014	0.869	2.150	5.020	14.62	84.46	0.704	5.617
IV	0.70	1.0	0.222	1.298	2.761	5.544	12.26	35.12	202.1	0.749	5.687
IV	1	0.1	-2.065	-0.822	-0.282	0.124	0.553	1.193	2.802	0.061	2.001
IV	1	0.5	-1.013	-0.237	0.207	0.630	1.175	2.117	4.726	0.263	2.027
IV	1	1.0	-0.249	0.316	0.775	1.313	2.100	3.566	7.808	0.387	2.072
V	0.30		1.352	2.916	6.074	13.67	36.94	144.9	1467	0.849	10.11
V	0.75		0.606	0.967	1.453	2.202	3.538	6.519	17.33	0.555	2.637
V	1		0.481	0.721	1.020	1.443	2.128	3.476	7.489	0.476	2.142
V	5		0.132	0.159	0.186	0.214	0.249	0.297	0.382	0.204	1.362
VI	0.30	0.3	0.007	0.073	0.308	1.000	3.244	13.79	143.1	0.865	10.22
VI	0.30	1.2	0.001	0.008	0.031	0.086	0.202	0.465	1.255	0.660	2.370
VI	0.30	2.4	0.000	0.003	0.013	0.036	0.083	0.180	0.428	0.630	2.026
VI	1	1	0.143	0.333	0.600	1.000	1.667	3.000	7.000	0.500	2.171
VI	1	4	0.034	0.075	0.125	0.189	0.278	0.414	0.682	0.325	1.456
VI	1	8	0.017	0.037	0.061	0.091	0.130	0.189	0.297	0.294	1.377
VII	0.30		-1.486	-0.923	-0.439	0.000	0.439	0.923	1.486	0.000	1.135
VII	0.75		-9.277	-2.197	-0.733	0.000	0.733	2.197	9.277	0.000	3.888
VII	1		-2.414	-1.000	-0.414	0.000	0.414	1.000	2.414	0.000	2.000
VII	1.5		-1.134	-0.577	-0.258	0.000	0.258	0.577	1.134	0.000	1.517

	I	III	IV	V	VI	VII
p_1	p	ρ	m	ρ	p	n
p_2	q		v		q	

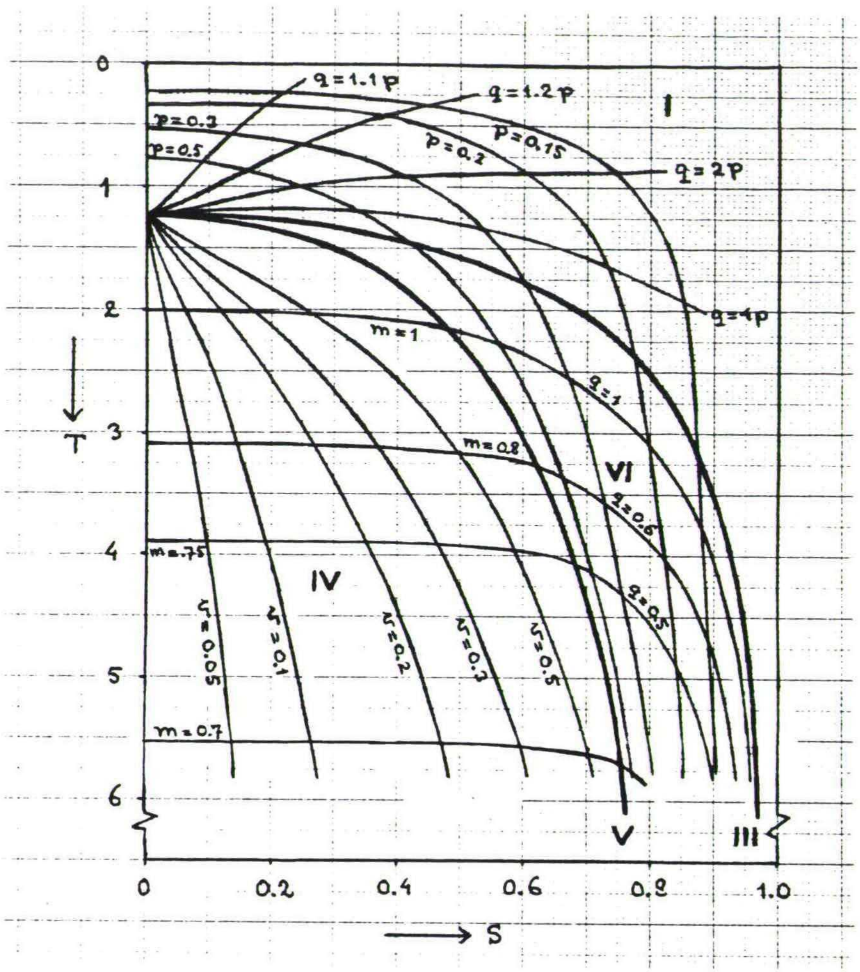
There appears to be one-one relation between the (location-scale free) Pearson distributions and pairs of (S,T)-values. So, just like the (β_1,β_2)-plane, the (S,T)-plane is subdivided into separate sets corresponding to the main types; the demarcation lines are given by the transition types. See Figure 4; compare it to Figure 3. The half-plane with $S < 0$ has been omitted.

Figure 4. The (S,T)-plane for the Pearson system.



Since within a given Pearson type the location-scale parameter is uniquely determined by (Q,R), all distributions in the Pearson system can be characterized by the quartet (Q,R,S,T). Hence, the empirical counterpart (q,r,s,t) of this quartet determines exactly one distribution within the Pearson system. Again, this gives a simple model for the empirical distribution, now based on quantile measures. However, the parameters of the corresponding Pearson model have to be found numerically. This can be done by trial-and-error, using the programs mentioned above. Another possibility is to develop a nomogram from which for given (S,T)-values the corresponding Pearson distribution can be read. In Figure 5 such a nomogram is sketched; of course, to attain numerical accuracy, a much more detailed nomogram is necessary.

Figure 5. Sketch of nomogram for the Pearson system.



In principle, there now are two ways to find a model within the Pearson system for a given frequency distribution. An interesting question is which model fits best; this question is discussed in some more detail in Section 6.

4. The Johnson system

Another subdivision of the (β_1, β_2)-plane was obtained by Johnson (1949). His system of distributions consists of three different types of transformations of a standard normal variable $\underline{z}$. Using

$$\underline{x} = (\underline{z}-\gamma)/\delta$$

for given constants γ and δ, these transformations are

$$\underline{y} = \varphi_L(\underline{x}) = \exp(\underline{x})$$

$$\underline{y} = \varphi_B(\underline{x}) = \exp(\underline{x})/[1 + \exp(\underline{x})]$$

$$\underline{y} = \varphi_U(\underline{x}) = [\exp(\underline{x}) - \exp(-\underline{x})]/2$$

Details of the resulting distributions are shown in Table 3.

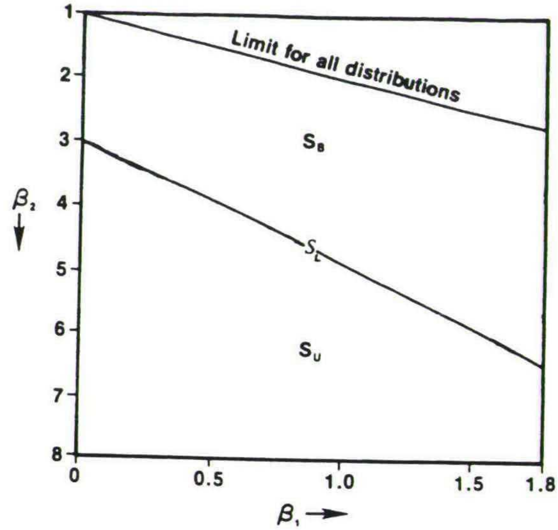
Table 3. Outline of the Johnson system.

Name	Type	Density	Range	Parameters
Lognormal	S_L	$\frac{\delta}{\sqrt{2\pi}} \frac{1}{y} \exp[-(\delta \log y)^2/2]$	$\mathbb{R}^+$	$\delta \in \mathbb{R}^+$
Bounded range	S_B	$\frac{\delta}{\sqrt{2\pi}} \frac{1}{y(1-y)} \exp[-\{\gamma+\delta \log(\frac{y}{1-y})\}^2/2]$	$[0,1]$	$\gamma \in \mathbb{R}$, $\delta \in \mathbb{R}^+$
Unbounded range	S_U	$\frac{\delta}{\sqrt{2\pi}} \frac{1}{\sqrt{1+y^2}} \exp[-\{\gamma+\delta \log(y+\sqrt{1+y^2})\}^2/2]$	$\mathbb{R}$	$\gamma \in \mathbb{R}$, $\delta \in \mathbb{R}^+$

For the lognormal distributions the - location - parameter γ has been deleted.

Like for the Pearson system, any distribution of the Johnson system has a unique pair of values for the measures β_1 and β_2. Figure 6 shows how the (β_1,β_2)-plane is split by the curve S_L in separate parts S_B and S_U. See for details Stuart & Ord (1987), p. 234 ff. The half-plane with $\beta_1 < 0$ has been omitted.

Figure 6. The (β_1,β_2)-plane for the Johnson system.



The quartet $(\bar{x},s^2,b_1,b_2)$ of empirical measures determines a unique distribution within the Johnson system, as was the case for the Pearson system.

5. Characterization of the Johnson system by quantile measures

The (S,T)-values of the transition type S_L arise as limits of the (S,T)-values of either S_U or S_B type distributions. E.g. for the latter type this is implied by the following property: if $\underline{x}_n \sim S_B(n,\delta)$, $\mathcal{L}(\underline{x}_n) \rightarrow S_L(\delta)$ for $n \rightarrow \infty$. So, in the (S,T)-plane the set defined by S_L is a smooth transition between the sets corresponding to S_U and S_B.

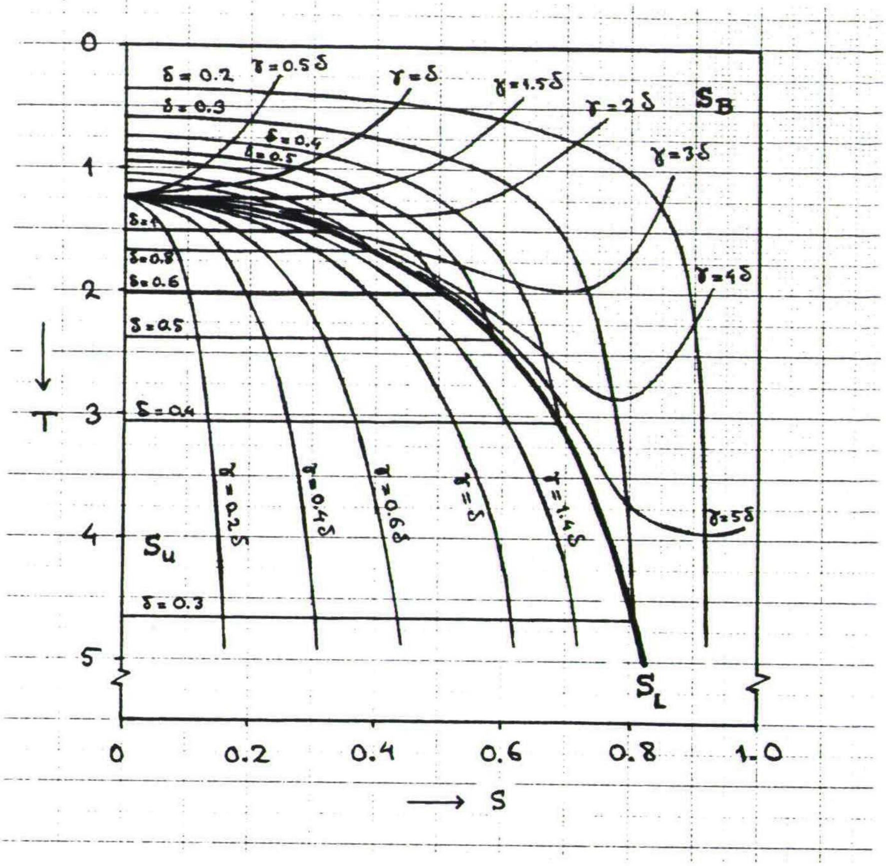
Since Johnson distributions are transformations of the standard normal, octiles are easily calculated. Table 4 gives a brief summary of the extensive tables in Wagemakers (1991).

Table 4. Octiles and (S,T)-values for the Johnson system.

Type	δ	γ	E_1	E_2	E_3	E_4	E_5	E_6	E_7	S	T
S_L	0.30		0.022	0.106	0.346	1.000	2.893	9.472	46.27	0.809	4.666
S_L	0.75		0.216	0.407	0.654	1.000	1.529	2.458	4.636	0.422	1.728
S_L	1		0.317	0.509	0.727	1.000	1.375	1.963	3.159	0.325	1.510
S_L	5		0.795	0.874	0.938	1.000	1.066	1.144	1.259	0.067	1.244
S_B	0.30	0	0.021	0.096	0.257	0.500	0.743	0.905	0.979	0.000	0.583
S_B	0.30	0.30	0.008	0.037	0.113	0.269	0.516	0.777	0.945	0.374	0.722
S_B	0.30	1.20	0.000	0.002	0.006	0.018	0.050	0.148	0.459	0.780	2.840
S_B	0.30	2.10	0.000	0.000	0.000	0.001	0.003	0.009	0.041	0.806	4.506
S_B	1	0	0.240	0.338	0.421	0.500	0.579	0.663	0.760	0.000	1.111
S_B	1	1	0.104	0.158	0.211	0.269	0.336	0.419	0.538	0.150	1.179
S_B	1	5	0.002	0.003	0.005	0.007	0.009	0.013	0.021	0.321	1.496
S_B	1	7	0.000	0.001	0.001	0.001	0.001	0.002	0.003	0.324	1.508
S_U	0.30	0.00	-23.12	-4.683	-1.273	0.000	1.273	4.683	23.12	0.000	4.666
S_U	0.30	0.12	-15.49	-3.096	-0.712	0.411	2.042	7.030	34.51	0.307	4.666
S_U	0.30	0.30	-8.482	-1.599	-0.062	1.175	3.868	12.85	62.88	0.616	4.666
S_U	0.30	0.60	-3.051	-0.251	1.082	3.627	10.66	34.99	171.0	0.780	4.666
S_U	1	0.0	-1.421	-0.727	-0.324	0.000	0.324	0.727	1.421	0.000	1.510
S_U	1	-0.5	-0.697	-0.175	0.182	0.521	0.913	1.464	2.508	0.150	1.510
S_U	1	-1.0	-0.151	0.231	0.735	1.175	1.735	2.574	4.236	0.248	1.510
S_U	1	-2.0	0.956	1.749	2.593	3.627	5.032	7.218	11.65	0.313	1.510

Again, from the empirical measures (q,r,s,t) a model can be found within the Johnson system. This may be done numerically or graphically, by means of a nomogram. Figure 7 sketches such a nomogram.

Figure 7. Sketch of nomogram for the Johnson system.



Again, their are two ways to find a suitable model within the Johnson system; an important question is whether the moment-based or the quantile-based approach is better.

6. Discussion and further research

In this paper an alternative method was developed to find a suitable model for an empirical frequency distribution within a given system of theoretical distributions. For this class of potential models both Pearson's and Johnson's system of distributions was considered. Our method is based on the four quantile measures

$$(Q,\ R,\ S,\ T)$$

for location, disperson, skewness and kurtosis; all of them can be calculated from the seven octiles.

Attention was concentrated on the behaviour of S and T; our main result is that both in Pearson's and in Johnson's system there is a one-one correspondence between the (location-scale free) distributions and the values of the pair (S,T).

An interesting next question is of course whether this quantile-based method gives a better fit than the classical approach, which is based on the moments

$$(\mu,\ \mu_2,\ \beta_1,\ \beta_2)$$

As a first step in answering this question, the limit distributions of the empirical measures $(\underline{s},\underline{t})$ and $(\underline{b}_1,\underline{b}_2)$ are being investigated. For the standard normal distribution we obtained the following results:

$$\sqrt{n}\begin{bmatrix}\underline{s}-S\\ \underline{t}-T\end{bmatrix} \overset{\mathcal{L}}{\rightarrow} N_2\left(\begin{bmatrix}0\\0\end{bmatrix},\ \begin{bmatrix}1.839 & 0\\ 0 & 3.153\end{bmatrix}\right)$$

$$\sqrt{n}\begin{bmatrix}\underline{b}_1-\beta_1\\ \underline{b}_2-\beta_2\end{bmatrix} \overset{\mathcal{L}}{\rightarrow} N_2\left(\begin{bmatrix}0\\0\end{bmatrix},\ \begin{bmatrix}6 & 0\\ 0 & 24\end{bmatrix}\right)$$

where $\overset{\mathcal{L}}{\rightarrow}$ denotes convergence in distribution for $n \rightarrow \infty$. Note that for N(0,1)

$$(S,T) = (0,\ 1.233)\ ,\qquad (\beta_1,\beta_2) = (0,\ 3)$$

holds. To check the (co)variances a simulation study was made. From N(0,1) 200 random sample of size n were drawn and for all of them the statistics

$$(s,\ t)\ ,\qquad (b_1,\ b_2)$$

were calculated. From the 200 replicated values the estimated variances and the covariance of each pair was found. Tables 5 and 6 present the results.

Table 5. Simulated means and (co)variance of $(\underline{s},\underline{t})$; 200 replicated samples from N(0,1).

n	Simulated value of $\sqrt{n}E(\underline{s})$	$\sqrt{n}E(\underline{t}-1.233)$	$nV(\underline{s})$	$nV(\underline{t})$	$nCov(\underline{s},\underline{t})$
50	-0.224	-0.149	1.887	3.940	0.256
100	0.103	0.395	1.746	3.378	-0.034
200	-0.039	0.395	1.931	3.487	-0.026
2000	-0.048	0.654	1.524	3.815	-0.283
∞	0	0	1.839	3.153	0

Table 6. Simulated means and (co)variance of $(\underline{b}_1,\underline{b}_2)$; 200 replicated samples from N(0,1).

n	Simulated value of $\sqrt{n}E(\underline{b}_1)$	$\sqrt{n}E(\underline{b}_2-3)$	$nV(\underline{b}_1)$	$nV(\underline{b}_2)$	$nCov(\underline{b}_1,\underline{b}_2)$
50	-0.240	-0.893	5.068	15.099	-0.068
100	0.095	-0.840	4.683	17.781	-0.035
200	-0.013	-0.461	5.453	21.181	1.185
2000	0.296	0.570	5.180	23.137	0.283
∞	0	0	6	24	0

A full report, with much more general results, is in preparation.

The simulation results appear to be in agreement with the theoretical values in the last lines of the two tables. Note that (S,T) can be estimated with a greater accuracy than (β_1,β_2). Of course, this does not imply that the quantile-based appoach is to be preferred. To admit such a conclusion, some measure of fit will have to be chosen and compared for both methods of modelling. We plan to make such a comparison in due course.

Apart from the Pearson and Johnson systems of distributions, other systems may be taken as the class of potential models. Interesting candidates are the Schmeiser-Deutsch (1978) system of distributions and Burr's system, cf. Stuart & Ord (1987), p. 242. A final question is how to select a suitable system to start with.

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The (R,b,Q)-model

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The (R,s,S)-model

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